

CONVERGENCE IN CAPACITY OF PLURISUBHARMONIC FUNCTIONS WITH GIVEN BOUNDARY VALUES

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ABSTRACT. In this paper, we study the convergence in the capacity of sequence of plurisubharmonic functions. As an application, we prove stability results for solutions of the complex Monge-Ampère equations.

1. INTRODUCTION

It is well-known that convergence in the sense of distributions of plurisubharmonic functions does not in general imply convergence of their Monge-Ampère measures. Therefore, it is important to find conditions on sequences of plurisubharmonic functions such that the corresponding Monge-Ampère measures are convergent in the weak* topology.

Bedford and Taylor [3] introduced and studied in 1982 the C_n -capacity of Borel sets. Xing [21] proved in 1996 that the complex Monge-Ampère operator is continuous under convergence of bounded plurisubharmonic functions in C_n -capacity. He gave a sufficient condition for the weak convergence of complex Monge-Ampère mass of bounded plurisubharmonic functions. Later, Xing [22] studied in 2008 the convergence in the C_n -capacity of a sequence of plurisubharmonic functions in the class $\mathcal{F}^a(\Omega)$. Hiep [15] studied in 2010 the convergence in C_n -capacity within the class $\mathcal{E}(\Omega)$. Recently, Cegrell [8] proved in 2012 that if a sequence of plurisubharmonic functions is bounded from below by a function from the Cegrell class $\mathcal{E}(\Omega)$ and convergent in C_{n-1} -capacity then the corresponding complex Monge-Ampère measures are convergent in the weak* topology.

The purpose of this paper is to study conditions on a sequence of plurisubharmonic functions which are equivalent to convergence in C_n -capacity. Our main result is the following theorem.

Main theorem. *Let Ω be a bounded hyperconvex domain in \mathbb{C}^n and let $f \in \mathcal{E}(\Omega)$, $w \in \mathcal{N}^a(\Omega, f)$ such that $\int_{\Omega} (-\rho)(dd^c w)^n < +\infty$ for some $\rho \in \mathcal{E}_0(\Omega)$. Assume that $\{u_j\} \subset \mathcal{N}^a(\Omega, f)$ such that $u_j \rightarrow u_0$ a.e. on Ω as $j \rightarrow +\infty$ and $u_j \geq w$ in Ω for all $j \geq 0$. Then, the following statements are equivalent.*

- (a) $u_j \rightarrow u_0$ in C_n -capacity in Ω ;
- (b) For every $a > 0$, we have

$$\lim_{j \rightarrow +\infty} \int_{\Omega} \max\left(\frac{u_j}{a}, \rho\right) (dd^c u_j)^n = \int_{\Omega} \max\left(\frac{u_0}{a}, \rho\right) (dd^c u_0)^n.$$

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(c) For every $a > 0$, we have

$$\lim_{j \rightarrow +\infty} \int_{\Omega} \left[\max \left(\frac{v_j}{a}, \rho \right) - \max \left(\frac{u_j}{a}, \rho \right) \right] (dd^c u_j)^n = 0,$$

where $v_j := (\sup_{k \geq j} u_k)^*$.

The paper is organized as follows. In Section 2 we recall some notions of pluripotential theory. Section 3 is devoted to the proof of the main theorem. In Section 4 we apply the main theorem to prove a stability result for the solutions of certain complex Monge-Ampère equations.

2. PRELIMINARIES

Some elements of pluripotential theory that will be used throughout the paper can be found in [1]-[22].

Definition 2.1. Let n be a positive integer. A bounded domain Ω in \mathbb{C}^n is called bounded hyperconvex domain if there exists a bounded plurisubharmonic function $\varphi : \Omega \rightarrow (-\infty, 0)$ such that the closure of the set $\{z \in \Omega : \varphi(z) < c\}$ is compact in Ω , for every $c \in (-\infty, 0)$.

We denote by $PSH(\Omega)$ the family of plurisubharmonic functions defined on Ω and $PSH^-(\Omega)$ denotes the set of negative plurisubharmonic functions on Ω . By $MPSH(\Omega)$ denotes the set of all maximal plurisubharmonic functions in Ω .

Definition 2.2. Let Ω be a bounded hyperconvex domain in \mathbb{C}^n . We say that a bounded, negative plurisubharmonic function φ in Ω belongs to $\mathcal{E}_0(\Omega)$ if $\{\varphi < -\varepsilon\} \Subset \Omega$ for all $\varepsilon > 0$ and $\int_{\Omega} (dd^c \varphi)^n < +\infty$.

Let $\mathcal{F}(\Omega)$ be the family of plurisubharmonic functions φ defined on Ω , such that there exists a decreasing sequence $\{\varphi_j\} \subset \mathcal{E}_0(\Omega)$ that converges pointwise to φ on Ω as $j \rightarrow +\infty$ and

$$\sup_j \int_{\Omega} (dd^c \varphi_j)^n < +\infty.$$

We denote by $\mathcal{E}(\Omega)$ the family of plurisubharmonic functions φ defined on Ω such that for every open set $G \Subset \Omega$ there exists a plurisubharmonic function $\psi \in \mathcal{F}(\Omega)$ satisfy $\psi = \varphi$ in G .

Let $u \in \mathcal{E}(\Omega)$ and let $\{\Omega_j\}$ be an increasing sequence of bounded hyperconvex domains such that $\Omega_j \Subset \Omega_j \Subset \Omega$ and $\bigcup_{j=1}^{+\infty} \Omega_j = \Omega$. Put

$$u^j := \sup\{\varphi \in PSH^-(\Omega) : \varphi \leq u \text{ in } \Omega \setminus \Omega_j\}$$

and $\mathcal{N}(\Omega) := \{u \in \mathcal{E}(\Omega) : u^j \nearrow 0 \text{ a.e. in } \Omega\}$.

Let $\mathcal{K} \in \{\mathcal{F}, \mathcal{N}, \mathcal{E}\}$. We denote by $\mathcal{K}^a(\Omega)$ the subclass of $\mathcal{K}(\Omega)$ such that the Monge-Ampère measure $(dd^c \cdot)^n$ vanishes on all pluripolar sets of Ω .

Let $f \in \mathcal{E}(\Omega)$ and $\mathcal{K} \in \{\mathcal{F}^a, \mathcal{N}^a, \mathcal{E}^a, \mathcal{F}, \mathcal{N}, \mathcal{E}\}$. Then we say that a plurisubharmonic function φ defined on Ω belongs to $\mathcal{K}(\Omega, f)$ if there exists a function $\psi \in \mathcal{K}(\Omega)$ such that

$$\psi + f \leq \varphi \leq f \text{ in } \Omega.$$

Now we will show that if $u \in \mathcal{N}^a(\Omega, f)$ then the pluripolar part of $(dd^c u)^n$ is carried by $\{f = -\infty\}$.

Proposition 2.3. *Let Ω be a bounded hyperconvex domain in \mathbb{C}^n . Assume that $f \in \mathcal{E}(\Omega)$ and $u \in \mathcal{N}^a(\Omega, f)$ such that $\int_{\Omega}(-\rho)(dd^c u)^n < +\infty$ for some $\rho \in \mathcal{E}_0(\Omega)$. Then*

$$1_{\{u=-\infty\}}(dd^c u)^n = 1_{\{f=-\infty\}}(dd^c f)^n \text{ in } \Omega.$$

Proof. Let $v \in \mathcal{F}^a(\Omega)$ such that $v + f \leq u \leq f$ in Ω . By Lemma 4.1 and Lemma 4.12 in [1] we have

$$\begin{aligned} 1_{\{f=-\infty\}}(dd^c f)^n &\leq 1_{\{u=-\infty\}}(dd^c u)^n \\ &\leq 1_{\{v+f=-\infty\}}(dd^c(v+f))^n = 1_{\{f=-\infty\}}(dd^c f)^n. \end{aligned}$$

It follows that

$$1_{\{u=-\infty\}}(dd^c u)^n = 1_{\{f=-\infty\}}(dd^c f)^n \text{ in } \Omega.$$

The proof is complete. \square

Proposition 2.4. *Let Ω be a bounded hyperconvex domain in \mathbb{C}^n . Let $f \in \mathcal{E}(\Omega)$ and $u \in \mathcal{N}^a(\Omega, f)$ such that $\int_{\Omega}(-\rho)(dd^c u)^n < +\infty$ for some $\rho \in \mathcal{E}_0(\Omega)$. Assume that $v \in \mathcal{E}(\Omega)$ such that $v \leq f$ and $(dd^c v)^n \geq (dd^c u)^n$ in Ω . Then $v \leq u$ on Ω .*

Proof. Since the measure $1_{\{u>-\infty\}}(dd^c u)^n$ vanishes on all pluripolar subsets of Ω , by Proposition 4.3 in [19] we get

$$(dd^c \max(u, v))^n \geq 1_{\{u>-\infty\}}(dd^c u)^n.$$

Hence,

$$1_{\{\max(u, v)>-\infty\}}(dd^c \max(u, v))^n \geq 1_{\{u>-\infty\}}(dd^c u)^n.$$

Moreover, by the hypotheses and Proposition 2.3 we have

$$1_{\{\max(u, v)=-\infty\}}(dd^c \max(u, v))^n = 1_{\{u=-\infty\}}(dd^c u)^n.$$

Hence, $(dd^c \max(u, v))^n \geq (dd^c u)^n$ in Ω . Therefore, from Theorem 3.6 in [1] it follows that $\max(u, v) = u$ in Ω . Thus, $v \leq u$ in Ω . The proof is complete. \square

3. PROOF OF THE MAIN THEOREM

In order to prove the main theorem, we need the following auxiliary lemmas.

Lemma 3.1. *Let Ω be a bounded hyperconvex domain in \mathbb{C}^n and let $f \in \mathcal{E}(\Omega)$. Assume that $\rho \in \mathcal{E}_0(\Omega)$ and $u \in \mathcal{N}^a(\Omega, f)$ such that $\int_{\Omega}(-\rho)(dd^c u)^n < +\infty$. Then for every $v \in \mathcal{E}^a(\Omega, f)$ and for every $\varphi \in \mathcal{E}_0(\Omega)$ with $\varphi \geq \rho$, we have*

$$\begin{aligned} \frac{1}{n!} \int_{\{u < v\}} (v - u)^n (dd^c \varphi)^n + \int_{\{u < v\}} -\varphi (dd^c v)^n \\ \leq \int_{\{u < v\}} -\varphi (dd^c u)^n. \end{aligned}$$

Proof. For $j \in \mathbb{N}^*$, put $v_j = \max(u, v - \frac{1}{j})$. Because $u \leq v_j \leq f$ in Ω , we have $v_j \in \mathcal{F}^a(\Omega, f)$. By Lemma 3.5 in [1] we have

$$\frac{1}{n!} \int_{\Omega} (v_j - u)^n (dd^c \varphi)^n + \int_{\Omega} -\varphi (dd^c v_j)^n \leq \int_{\Omega} -\varphi (dd^c u)^n.$$

By Theorem 4.1 in [19] we have $v_j = v - \frac{1}{j}$ in $\{u < v_j\}$. Hence,

$$\begin{aligned} \frac{1}{n!} \int_{\{u < v_j\}} (v_j - u)^n (dd^c \varphi)^n + \int_{\{u < v_j\}} -\varphi (dd^c v)^n \\ = \frac{1}{n!} \int_{\{u < v_j\}} (v_j - u)^n (dd^c \varphi)^n + \int_{\{u < v_j\}} -\varphi (dd^c v_j)^n \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{n!} \int_{\Omega} (v_j - u)^n (dd^c \varphi)^n + \int_{\{u < v_j\}} -\varphi (dd^c v_j)^n \\
&\leq \frac{1}{n!} \int_{\Omega} (v_j - u)^n (dd^c \varphi)^n + \int_{\Omega} -\varphi (dd^c v_j)^n - \int_{\{u = v_j\}} -\varphi (dd^c v_j)^n \\
&\leq \int_{\Omega} -\varphi (dd^c u)^n - \int_{\{u \geq v\}} -\varphi (dd^c v_j)^n.
\end{aligned}$$

Now, since $u = v_j$ in $\{u > v - \frac{1}{j}\}$ so by Theorem 4.1 in [19] imply that

$$(dd^c u)^n = (dd^c v_j)^n \text{ in } \{u \geq v\} \cap \{u > -\infty\}.$$

Moreover, by Proposition 2.3 we have

$$1_{\{u = -\infty\}} (dd^c u)^n = 1_{\{v_j = -\infty\}} (dd^c v_j)^n = 1_{\{f = -\infty\}} (dd^c f)^n \text{ in } \Omega.$$

Hence, we obtain that

$$(dd^c u)^n = (dd^c v_j)^n \text{ in } \{u \geq v\}.$$

Therefore,

$$\begin{aligned}
&\frac{1}{n!} \int_{\{u < v_j\}} (v_j - u)^n (dd^c \varphi)^n + \int_{\{u < v_j\}} -\varphi (dd^c v)^n \\
&\leq \int_{\Omega} -\varphi (dd^c u)^n - \int_{\{u \geq v\}} -\varphi (dd^c u)^n \\
&= \int_{\{u < v\}} -\varphi (dd^c u)^n.
\end{aligned}$$

Let $j \rightarrow +\infty$ we obtain that

$$\frac{1}{n!} \int_{\{u < v\}} (v - u)^n (dd^c \varphi)^n + \int_{\{u < v\}} -\varphi (dd^c v)^n \leq \int_{\{u < v\}} -\varphi (dd^c u)^n.$$

The proof is complete. \square

Lemma 3.2. *Let Ω be a bounded hyperconvex domain in \mathbb{C}^n and let $\{u_j\} \subset \mathcal{E}^a(\Omega)$ such that $u_j \geq u_1$ for every $j \geq 1$ and $u_j \rightarrow u_0$ in C_n -capacity in Ω . Assume that $\{\varphi_j^k\}$, $k = 1, 2$ are sequences of uniformly bounded plurisubharmonic functions in Ω which converges weakly to a plurisubharmonic function φ_0^k in Ω . Then $\varphi_j^1 \varphi_j^2 (dd^c u_j)^n \rightarrow \varphi_0^1 \varphi_0^2 (dd^c u_0)^n$ weakly as $j \rightarrow +\infty$.*

Proof. Without loss of generality we can assume that $u_j \in \mathcal{F}^a(\Omega)$ and $-1 \leq \varphi_j^k \leq 0$ in Ω for all $j \geq 0$, $k = 1, 2$. Put

$$\psi_j^1 = \frac{(\varphi_j^1 + \varphi_j^2 + 2)^2 + 4}{2}, \quad \psi_j^2 = \frac{(\varphi_j^1 + 2)^2}{2} \text{ and } \psi_j^3 = \frac{(\varphi_j^2 + 2)^2}{2}.$$

It is clear that $\psi_j^k \in PSH(\Omega)$, $0 \leq \psi_j^k \leq 4$ and $\psi_j^k \rightarrow \psi_0^k$ weakly in Ω as $j \rightarrow +\infty$, $k = 1, 2, 3$. Since $\varphi_j^1 \varphi_j^2 = \psi_j^1 - \psi_j^2 - \psi_j^3$ in Ω we obtain by Theorem 3.4 in [22] that

$$\begin{aligned}
\varphi_j^1 \varphi_j^2 (dd^c u_j)^n &= \psi_j^1 (dd^c u_j)^n - \psi_j^2 (dd^c u_j)^n - \psi_j^3 (dd^c u_j)^n \\
&\rightarrow \psi_0^1 (dd^c u_0)^n - \psi_0^2 (dd^c u_0)^n - \psi_0^3 (dd^c u_0)^n = \varphi_0^1 \varphi_0^2 (dd^c u_0)^n
\end{aligned}$$

weakly in Ω as $j \rightarrow +\infty$. The proof is complete. \square

Proof of the main theorem. Without loss of generality we can assume that $f < 0$ and $-1 \leq \rho \leq 0$ in Ω .

(a) \Rightarrow (b). Fix $a > 0$. Put

$$\varphi_j := \max\left(\frac{u_j}{a}, \rho\right).$$

Because

$$0 \leq \sup_j \int_{\Omega} -\varphi_j (dd^c u_j)^n \leq \int_{\Omega} -\rho (dd^c w)^n < +\infty,$$

it remains to prove that there exists a subsequence $\{u_{j_k}\}$ of sequence $\{u_j\}$ such that

$$\lim_{k \rightarrow +\infty} \int_{\Omega} \varphi_{j_k} (dd^c u_{j_k})^n = \int_{\Omega} \varphi_0 (dd^c u_0)^n.$$

First we claim that there exists an increasing sequence $\{j_k\} \subset \mathbb{N}^*$ such that

$$\lim_{k \rightarrow +\infty} \int_{\Omega} \varphi_{j_k} \max\left(1 + \frac{u_{j_k}}{k}, 0\right) (dd^c u_{j_k})^n = \int_{\{u_0 > -\infty\}} \varphi_0 (dd^c u_0)^n. \quad (3.1)$$

Indeed, let $\chi_k \in \mathcal{C}_0^\infty(\Omega)$ such that $0 \leq \chi_k \leq \chi_{k+1} \leq 1$ in Ω , $\{\rho \leq -\frac{1}{k}\} \Subset \{\chi_k = 1\}$ and

$$\int_{\{\chi_k < 1\}} (-\rho) (dd^c u_0)^n \leq \frac{1}{k}.$$

Since $u_j \rightarrow u_0$ in C_n -capacity in Ω as $j \rightarrow +\infty$, so $\max(u_j, -k) \rightarrow \max(u_0, -k)$ in C_n -capacity as $j \rightarrow +\infty$. By Lemma 3.2 we have

$$\varphi_j \max\left(1 + \frac{u_j}{k}, 0\right) (dd^c \max(u_j, -k))^n \rightarrow \varphi_0 \max\left(1 + \frac{u_0}{k}, 0\right) (dd^c \max(u_0, -k))^n$$

weakly in Ω as $j \rightarrow +\infty$. Hence, by Theorem 4.1 in [19] we get

$$\begin{aligned} & \lim_{j \rightarrow +\infty} \int_{\Omega} \chi_k \varphi_j \max\left(1 + \frac{u_j}{k}, 0\right) (dd^c u_j)^n \\ &= \lim_{j \rightarrow +\infty} \int_{\Omega} \chi_k \varphi_j \max\left(1 + \frac{u_j}{k}, 0\right) (dd^c \max(u_j, -k))^n \\ &= \int_{\Omega} \chi_k \varphi_0 \max\left(1 + \frac{u_0}{k}, 0\right) (dd^c \max(u_0, -k))^n \\ &= \int_{\Omega} \chi_k \varphi_0 \max\left(1 + \frac{u_0}{k}, 0\right) (dd^c u_0)^n. \end{aligned}$$

Because $\chi_k \max\left(1 + \frac{u_0}{k}, 0\right) \nearrow 1_{\{u_0 > -\infty\}}$ as $k \rightarrow +\infty$ in Ω , we have

$$\lim_{k \rightarrow +\infty} \int_{\Omega} \chi_k \varphi_0 \max\left(1 + \frac{u_0}{k}, 0\right) (dd^c u_0)^n = \int_{\{u_0 > -\infty\}} \varphi_0 (dd^c u_0)^n.$$

Therefore, there exists an increasing sequence $\{j_k\} \subset \mathbb{N}^*$ such that

$$\lim_{k \rightarrow +\infty} \int_{\Omega} \chi_k \varphi_{j_k} \max\left(1 + \frac{u_{j_k}}{k}, 0\right) (dd^c u_{j_k})^n = \int_{\{u_0 > -\infty\}} \varphi_0 (dd^c u_0)^n. \quad (3.2)$$

Now, fix $k_0 \in \mathbb{N}^*$. By the proof of the theorem in [8] (see (3.1) in [8]) we have

$$\begin{aligned}
& \liminf_{k \rightarrow +\infty} \int_{\Omega} (1 - \chi_k) \varphi_{j_k} \max \left(1 + \frac{u_{j_k}}{k}, 0 \right) (dd^c u_{j_k})^n \\
& \geq \liminf_{k \rightarrow +\infty} \int_{\Omega} (1 - \chi_k) \varphi_{j_k} (dd^c u_{j_k})^n \geq \liminf_{k \rightarrow +\infty} \int_{\Omega} (1 - \chi_{k_0}) \rho (dd^c u_{j_k})^n \\
& = \liminf_{k \rightarrow +\infty} \left[\int_{\Omega} \rho (dd^c u_{j_k})^n - \int_{\Omega} \chi_{k_0} \rho (dd^c u_{j_k})^n \right] \\
& = \int_{\Omega} \rho (dd^c u_0)^n - \int_{\Omega} \chi_{k_0} \rho (dd^c u_0)^n \\
& \geq \int_{\{\chi_{k_0} < 1\}} \rho (dd^c u_0)^n \geq -\frac{1}{k_0}.
\end{aligned} \tag{3.3}$$

Combining this with (3.2) we arrive at

$$\begin{aligned}
& \lim_{k \rightarrow +\infty} \int_{\Omega} \varphi_{j_k} \max \left(1 + \frac{u_{j_k}}{k}, 0 \right) (dd^c u_{j_k})^n \\
& = \lim_{k \rightarrow +\infty} \int_{\Omega} \chi_k \varphi_{j_k} \max \left(1 + \frac{u_{j_k}}{k}, 0 \right) (dd^c u_{j_k})^n \\
& = \int_{\{u_0 > -\infty\}} \varphi_0 (dd^c u_0)^n.
\end{aligned}$$

This proves the claim.

The measure $1_{\{u_{j_k} > -\infty\}} \varphi_{j_k} \max \left(\frac{u_{j_k}}{k}, -1 \right) (dd^c u_{j_k})^n$ vanishes on all pluripolar subset of Ω , hence by Lemma 5.14 in [6] there exists $h_k \in \mathcal{F}^a(\Omega)$ such that

$$(dd^c h_k)^n = 1_{\{u_{j_k} > -\infty\}} \varphi_{j_k} \max \left(\frac{u_{j_k}}{k}, -1 \right) (dd^c u_{j_k})^n.$$

Because $(dd^c h_k)^n \leq (dd^c u_{j_k})^n$ in Ω and the measure $(dd^c h_k)^n$ vanishes on all pluripolar subset of Ω , from Corollary 3.2 in [1] we have

$$h_k \geq u_{j_k} \geq w \text{ in } \Omega.$$

We claim that $h_k \rightarrow 0$ in C_n -capacity in Ω . Indeed, let $\delta > 0$ and $\psi \in PSH(\Omega)$ with $-1 \leq \psi \leq 0$. By Theorem 3.1 in [1] we have

$$\begin{aligned}
\int_{\{h_k < -\delta\}} (dd^c \psi)^n & \leq \int_{\{h_k < \delta \psi\}} (dd^c \psi)^n \leq \frac{1}{\delta^n} \int_{\{h_k < \delta \psi\}} (dd^c h_k)^n \\
& \leq \frac{1}{\delta^n} \int_{\{u_{j_k} > -\infty\}} \varphi_{j_k} \max \left(\frac{u_{j_k}}{k}, -1 \right) (dd^c u_{j_k})^n \\
& \leq -\frac{1}{\delta^n} \int_{\{u_{j_k} > -\infty\}} \max \left(\frac{u_{j_k}}{k}, \rho \right) (dd^c u_{j_k})^n.
\end{aligned}$$

Therefore, by Lemma 3.3 in [1] and Proposition 2.3 we obtain that

$$\begin{aligned}
\int_{\{h_k < -\delta\}} (dd^c \psi)^n & \leq -\frac{1}{\delta^n} \int_{\Omega} \max \left(\frac{u_{j_k}}{k}, \rho \right) (dd^c u_{j_k})^n + \frac{1}{\delta^n} \int_{\{u_{j_k} = -\infty\}} \rho (dd^c u_{j_k})^n \\
& \leq -\frac{1}{\delta^n} \int_{\Omega} \max \left(\frac{w}{k}, \rho \right) (dd^c w)^n + \frac{1}{\delta^n} \int_{\{w = -\infty\}} \rho (dd^c w)^n \\
& = -\frac{1}{\delta^n} \int_{\{w > -\infty\}} \max \left(\frac{w}{k}, \rho \right) (dd^c w)^n.
\end{aligned}$$

It follows that

$$C_n(\{h_k < -\delta\}) \leq -\frac{1}{\delta^n} \int_{\{w > -\infty\}} \max\left(\frac{w}{k}, \rho\right) (dd^c w)^n.$$

Hence, we get

$$\lim_{k \rightarrow +\infty} C_n(\{h_k < -\delta\}) = 0,$$

for every $\delta > 0$. Thus, $h_k \rightarrow 0$ in C_n -capacity in Ω as $k \rightarrow +\infty$. This proves the claim, and therefore, by (3.3) and the Theorem in [8] we have

$$\begin{aligned} 0 &\leq \limsup_{k \rightarrow +\infty} \int_{\{u_{j_k} > -\infty\}} \varphi_{j_k} \max\left(\frac{u_{j_k}}{k}, -1\right) (dd^c u_{j_k})^n \\ &\leq \limsup_{k \rightarrow +\infty} \int_{\Omega} (1 - \chi_{k_0})(-\rho)(dd^c u_{j_k})^n + \limsup_{k \rightarrow +\infty} \int_{\Omega} \chi_{k_0}(dd^c h_k)^n \\ &\leq \frac{1}{k_0}, \end{aligned}$$

for all $k_0 \in \mathbb{N}^*$. Thus,

$$\lim_{k \rightarrow +\infty} \int_{\{u_{j_k} > -\infty\}} \varphi_{j_k} \max\left(\frac{u_{j_k}}{k}, -1\right) (dd^c u_{j_k})^n = 0.$$

Combining this with (3.1) we arrive at

$$\lim_{k \rightarrow +\infty} \int_{\{u_{j_k} > -\infty\}} \varphi_{j_k} (dd^c u_{j_k})^n = \int_{\{u_0 > -\infty\}} \varphi_0 (dd^c u_0)^n.$$

Moreover, by Proposition 2.3, we have

$$\int_{\{u_{j_k} = -\infty\}} \varphi_{j_k} (dd^c u_{j_k})^n = \int_{\{f = -\infty\}} \rho (dd^c f)^n = \int_{\{u_0 = -\infty\}} \varphi_0 (dd^c u_0)^n.$$

Hence, we get

$$\lim_{k \rightarrow +\infty} \int_{\Omega} \varphi_{j_k} (dd^c u_{j_k})^n = \int_{\Omega} \varphi_0 (dd^c u_0)^n.$$

(b) \Rightarrow (c). Fix $a > 0$. Since $u_j \rightarrow u_0$ a.e. in Ω as $j \rightarrow +\infty$ so $v_j \searrow u_0$ as $j \nearrow +\infty$. Hence, $v_j \rightarrow u_0$ in C_n -capacity in Ω . Therefore, by the proof of (a) \Rightarrow (b) and Lemma 3.3 in [1], we have

$$\begin{aligned} \int_{\Omega} \max\left(\frac{u_0}{a}, \rho\right) (dd^c u_0)^n &= \lim_{j \rightarrow +\infty} \int_{\Omega} \max\left(\frac{v_j}{a}, \rho\right) (dd^c v_j)^n \\ &\geq \lim_{j \rightarrow +\infty} \int_{\Omega} \max\left(\frac{v_j}{a}, \rho\right) (dd^c u_j)^n \\ &\geq \lim_{j \rightarrow +\infty} \int_{\Omega} \max\left(\frac{u_j}{a}, \rho\right) (dd^c u_j)^n \\ &= \int_{\Omega} \max\left(\frac{u_0}{a}, \rho\right) (dd^c u_0)^n. \end{aligned}$$

It follows that

$$\lim_{j \rightarrow +\infty} \int_{\Omega} \max\left(\frac{v_j}{a}, \rho\right) (dd^c u_j)^n = \int_{\Omega} \max\left(\frac{u_0}{a}, \rho\right) (dd^c u_0)^n.$$

Therefore, we obtain that

$$\lim_{j \rightarrow +\infty} \int_{\Omega} \left[\max\left(\frac{v_j}{a}, \rho\right) - \max\left(\frac{u_j}{a}, \rho\right) \right] (dd^c u_j)^n = 0.$$

(c) \Rightarrow (a). Because $v_j \searrow u_0$ in Ω as $j \nearrow +\infty$, we get $v_j \rightarrow u_0$ in C_n -capacity in Ω . Hence, it is sufficient to prove that $v_j - u_j \rightarrow 0$ in C_n -capacity in Ω . Let K be a compact subset of Ω and let $\varepsilon, \delta > 0$. Without loss of generality we can assume that $K \subseteq \{\rho = -1\}$. Choose $\chi \in \mathcal{C}_0^\infty(\Omega)$ and $a > b > 1$ such that $0 \leq \chi \leq 1$, $\{\rho \leq -\varepsilon\} \subseteq \{\chi = 1\}$, $\{\chi \neq 0\} \subset \{a\rho < -b\}$ and

$$\frac{a}{b} \int_{\{w > -\infty\}} -\max\left(\frac{w}{a}, \rho\right) (dd^c w)^n < \varepsilon. \quad (3.4)$$

Let $\psi_j \in \mathcal{E}_0(\Omega)$ with $\psi_j \geq \rho$ such that

$$C_n(K \cap \{v_j - u_j > 2\delta\}) < \int_{K \cap \{v_j - u_j > 2\delta\}} (dd^c \psi_j)^n + \varepsilon. \quad (3.5)$$

Note that $u_j \leq v_j$ in Ω for all $j \geq 1$. From the hypotheses we have

$$\begin{aligned} 0 &\leq \limsup_{j \rightarrow +\infty} \int_{\{u_j < v_j - \delta\} \cap \{u_j > -b\}} \chi (dd^c u_j)^n \\ &\leq \frac{1}{\delta} \limsup_{j \rightarrow +\infty} \int_{\{u_j < v_j - \delta\} \cap \{u_j > -b\}} \chi (v_j - u_j) (dd^c u_j)^n \\ &\leq \frac{a}{\delta} \limsup_{j \rightarrow +\infty} \int_{\Omega} \left[\max\left(\frac{v_j}{a}, \rho\right) - \max\left(\frac{u_j}{a}, \rho\right) \right] (dd^c u_j)^n \\ &= 0. \end{aligned}$$

It follow that

$$\lim_{j \rightarrow +\infty} \int_{\{u_j < v_j - \delta\} \cap \{u_j > -b\}} \chi (dd^c u_j)^n = 0.$$

By Lemma 3.3 in [1] and Proposition 2.3 we have

$$\begin{aligned} &\limsup_{j \rightarrow +\infty} \int_{\{u_j < v_j - \delta\} \cap \{u_j > -\infty\}} \chi (dd^c u_j)^n \\ &= \limsup_{j \rightarrow +\infty} \int_{\{u_j < v_j - \delta\} \cap \{-\infty < u_j \leq -b\}} \chi (dd^c u_j)^n \\ &\leq \limsup_{j \rightarrow +\infty} \int_{\{u_j > -\infty\}} -\max\left(\frac{u_j}{b}, \frac{a\rho}{b}\right) (dd^c u_j)^n \\ &\leq \frac{a}{b} \limsup_{j \rightarrow +\infty} \int_{\Omega} -\max\left(\frac{u_j}{a}, \rho\right) (dd^c u_j)^n + \frac{a}{b} \int_{\{u_j = -\infty\}} \max\left(\frac{u_j}{a}, \rho\right) (dd^c u_j)^n \\ &\leq \frac{a}{b} \int_{\Omega} -\max\left(\frac{w}{a}, \rho\right) (dd^c w)^n + \frac{a}{b} \int_{\{w = -\infty\}} \max\left(\frac{w}{a}, \rho\right) (dd^c w)^n \\ &= \frac{a}{b} \int_{\{w > -\infty\}} -\max\left(\frac{w}{a}, \rho\right) (dd^c w)^n. \end{aligned}$$

Therefore, by (3.4) we get

$$\limsup_{j \rightarrow +\infty} \int_{\{u_j < v_j - \delta\} \cap \{u_j > -\infty\}} \chi (dd^c u_j)^n < \varepsilon. \quad (3.6)$$

Now, by Proposition 2.3 and Lemma 3.1 we have

$$\begin{aligned} \int_{K \cap \{v_j - u_j > 2\delta\}} (dd^c \psi_j)^n &\leq \int_{\{u_j < v_j - 2\delta\}} (dd^c \psi_j)^n \\ &\leq \frac{1}{\delta^n} \int_{\{u_j < v_j - 2\delta\}} (v_j - \delta - u_j)^n (dd^c \psi_j)^n \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\delta^n} \int_{\{u_j < v_j - \delta\}} (v_j - \delta - u_j)^n (dd^c \psi_j)^n \\
&\leq \frac{n!}{\delta^n} \int_{\{u_j < v_j - \delta\} \cap \{u_j > -\infty\}} -\psi_j (dd^c u_j)^n.
\end{aligned}$$

Hence, from (3.6) we obtain that

$$\begin{aligned}
\limsup_{j \rightarrow +\infty} \int_{K \cap \{v_j - u_j > 2\delta\}} (dd^c \psi_j)^n &\leq \frac{n!}{\delta^n} \limsup_{j \rightarrow +\infty} \int_{\{u_j < v_j - \delta\} \cap \{u_j > -\infty\}} -\psi_j (dd^c u_j)^n \\
&\leq \frac{n!}{\delta^n} \limsup_{j \rightarrow +\infty} \int_{\Omega} -\psi_j (1 - \chi) (dd^c u_j)^n \\
&\quad + \frac{n!}{\delta^n} \limsup_{j \rightarrow +\infty} \int_{\{u_j < v_j - \delta\} \cap \{u_j > -\infty\}} \chi (dd^c u_j)^n \\
&\leq \frac{n!}{\delta^n} \limsup_{j \rightarrow +\infty} \int_{\{\rho > -\varepsilon\}} -\rho (dd^c u_j)^n + \frac{n! \varepsilon}{\delta^n} \\
&\leq \frac{n!}{\delta^n} \limsup_{j \rightarrow +\infty} \int_{\Omega} -\max(\rho, -\varepsilon) (dd^c u_j)^n + \frac{n! \varepsilon}{\delta^n} \\
&\leq \frac{n!}{\delta^n} \int_{\Omega} -\max(\rho, -\varepsilon) (dd^c w)^n + \frac{n! \varepsilon}{\delta^n}.
\end{aligned}$$

Combining this with (3.5) we get

$$\limsup_{j \rightarrow +\infty} C_n(\{K \cap \{v_j - u_j > 2\delta\}\}) \leq \frac{n!}{\delta^n} \int_{\Omega} -\max(\rho, -\varepsilon) (dd^c w)^n + \left(\frac{n!}{\delta^n} + 1\right) \varepsilon.$$

Let $\varepsilon \searrow 0$ we obtain that

$$\lim_{j \rightarrow +\infty} C_n(\{K \cap \{v_j - u_j > 2\delta\}\}) = 0.$$

Thus, $v_j - u_j \rightarrow 0$ in C_n -capacity in Ω . The proof is complete. \square

4. APPLICATION

In this section, we prove a generalization of Cegrell and Kołodziej's stability theorem from [9]. First, we need the following.

Lemma 4.1. *Let Ω be a bounded hyperconvex domain in \mathbb{C}^n and let $f \in \mathcal{E}(\Omega)$, $w \in \mathcal{N}^a(\Omega, f)$ such that $\int_{\Omega} (-\rho) (dd^c w)^n < +\infty$ for some $\rho \in \mathcal{E}_0(\Omega)$. Then for every nonnegative Borel measures μ in Ω such that*

$$(dd^c f)^n \leq \mu \leq (dd^c w)^n,$$

there exists a unique $u \in \mathcal{N}^a(\Omega, f)$ such that $u \geq w$ and $(dd^c u)^n = \mu$ in Ω .

Proof. The uniqueness imply from Proposition 2.4. From the hypotheses and Proposition 2.3 we have

$$1_{\{w = -\infty\}} \mu = 1_{\{f = -\infty\}} (dd^c f)^n \text{ in } \Omega.$$

Let $\{\Omega_j\}$ be a sequence of bounded hyperconvex domains such that $\Omega_j \Subset \Omega_{j+1} \Subset \Omega$ and $\Omega = \bigcup_{j=1}^{+\infty} \Omega_j$. Because the measure $1_{\{w > -\infty\}} \mu$ vanishes on all pluripolar subsets of Ω , applying Proposition 5.1 in [14] we see that there are $u_j \in \mathcal{N}^a(\Omega_j, f)$ such that

$$(dd^c u_j)^n = 1_{\Omega_j \cap \{w > -\infty\}} \mu + 1_{\Omega_j \cap \{f = -\infty\}} (dd^c f)^n = \mu \text{ in } \Omega_j.$$

By Proposition 2.4 we have $w \leq u_{j+1} \leq u_j \leq f$ on Ω_j . Put $u := \lim_{j \rightarrow +\infty} u_j$. Then $w \leq u \leq f$ and $(dd^c u)^n = \mu$ in Ω . Moreover, since $w \in \mathcal{N}^a(\Omega, f)$, we get $u \in \mathcal{N}^a(\Omega, f)$. The proof is complete. \square

Proposition 4.2. *Let Ω be a bounded hyperconvex domain in \mathbb{C}^n and let $f \in \mathcal{E}(\Omega)$. Assume that $w \in \mathcal{N}^a(\Omega, f)$ such that $\int_{\Omega} (-\rho)(dd^c w)^n < +\infty$ for some $\rho \in \mathcal{E}_0(\Omega)$. Then for every sequence of nonnegative Borel measures $\{\mu_j\}$ that converges weakly to a non-negative Borel measure μ_0 in Ω and satisfies*

$$(dd^c f)^n \leq \mu_j \leq (dd^c w)^n \text{ for all } j \geq 0,$$

there exist unique $u_j \in \mathcal{N}^a(\Omega, f)$ such that $u_j \geq w$, $(dd^c u_j)^n = \mu_j$ and $u_j \rightarrow u_0$ in C_n -capacity in Ω .

Proof. By Lemma 4.1 there exist unique $u_j \in \mathcal{N}^a(\Omega, f)$ such that $u_j \geq w$ and $(dd^c u_j)^n = \mu_j$ in Ω . Since $u_j \geq w$, the sequence $\{u_j\}$ is compact in $L^1_{loc}(\Omega)$. Let u be a cluster point and let $\{u_{j_k}\}$ be a subsequence of the sequence $\{u_j\}$ such that $u_{j_k} \rightarrow u$ a.e. in Ω . Put $v_k := (\sup_{l \geq k} u_{j_l})^*$. We claim that

$$\lim_{k \rightarrow +\infty} \int_{\Omega} \left[\max\left(\frac{v_k}{a}, \rho\right) - \max\left(\frac{u_{j_k}}{a}, \rho\right) \right] (dd^c u_{j_k})^n = 0, \quad (4.1)$$

for every $a > 0$. Indeed, let $\varepsilon > 0$. Choose $\chi \in \mathcal{C}_0^\infty(\Omega)$ such that $0 \leq \chi \leq 1$ and $\{\chi = 1\} \subset \{\rho < -\varepsilon\}$. By Proposition 2.3 we have that the measure $1_{\{f > -\infty\}} \chi (dd^c w)^n$ vanishes on all pluripolar subsets of Ω . By Lemma 3.1 in [8] we get

$$\begin{aligned} 0 &\leq \limsup_{k \rightarrow +\infty} \int_{\Omega} \left[\max\left(\frac{v_k}{a}, \rho\right) - \max\left(\frac{u_{j_k}}{a}, \rho\right) \right] \chi (dd^c u_{j_k})^n \\ &= \limsup_{k \rightarrow +\infty} \int_{\Omega} \left[\max\left(\frac{v_k}{a}, \rho\right) - \max\left(\frac{u_{j_k}}{a}, \rho\right) \right] 1_{\{f > -\infty\}} \chi (dd^c u_{j_k})^n \\ &\leq \limsup_{k \rightarrow +\infty} \int_{\Omega} \left[\max\left(\frac{v_k}{a}, \rho\right) - \max\left(\frac{u_{j_k}}{a}, \rho\right) \right] 1_{\{f > -\infty\}} \chi (dd^c w)^n \\ &= \int_{\Omega} \left[\max\left(\frac{u}{a}, \rho\right) - \max\left(\frac{u}{a}, \rho\right) \right] 1_{\{f > -\infty\}} \chi (dd^c w)^n \\ &= 0. \end{aligned}$$

It follows that

$$\begin{aligned} 0 &\leq \limsup_{k \rightarrow +\infty} \int_{\Omega} \left[\max\left(\frac{v_k}{a}, \rho\right) - \max\left(\frac{u_{j_k}}{a}, \rho\right) \right] (dd^c u_{j_k})^n \\ &\leq \limsup_{k \rightarrow +\infty} \int_{\Omega} \left[\max\left(\frac{v_k}{a}, \rho\right) - \max\left(\frac{u_{j_k}}{a}, \rho\right) \right] (1 - \chi) (dd^c u_{j_k})^n \\ &\leq \limsup_{k \rightarrow +\infty} \int_{\{\rho \geq -\varepsilon\}} \left[-\max\left(\frac{v_k}{a}, \rho\right) - \max\left(\frac{u_{j_k}}{a}, \rho\right) \right] (dd^c u_{j_k})^n \\ &\leq 2 \int_{\Omega} -\max(\rho, -\varepsilon) (dd^c w)^n. \end{aligned}$$

Let $\varepsilon \searrow 0$ we obtain (4.1). This proves the claim, and therefore, by the main theorem we get $u_{j_k} \rightarrow u$ in C_n -capacity in Ω as $k \rightarrow +\infty$. Hence, by [8] we have $(dd^c u)^n = \mu_0$ in Ω . It is clear that $u \in \mathcal{N}^a(\Omega, f)$. From the uniqueness of u_0 we get $u = u_0$. Thus, $u_{j_k} \rightarrow u_0$ a.e. in Ω . It follows that $u_j \rightarrow u_0$ a.e. in Ω . Similarly, we get

$$\lim_{j \rightarrow +\infty} \int_{\Omega} \left[\max\left(\frac{v_j}{a}, \rho\right) - \max\left(\frac{u_j}{a}, \rho\right) \right] (dd^c u_j)^n = 0,$$

for every $a > 0$, where $v_j := (\sup_{k \geq j} u_k)^*$. Now, again by the main theorem we get $u_j \rightarrow u_0$ in C_n -capacity in Ω . The proof is complete. \square

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